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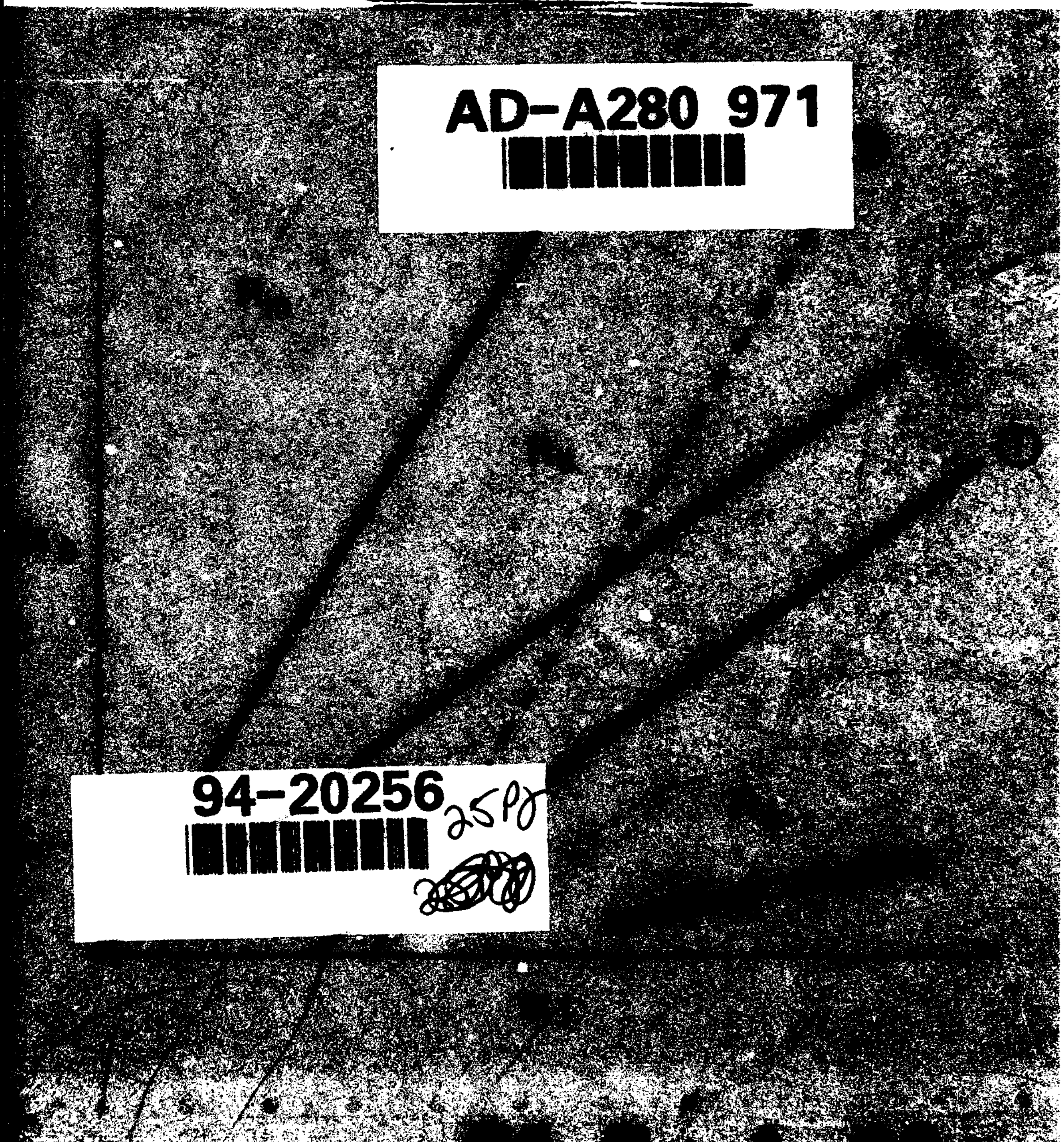
Traveling Wave Solutions to the Problem of Quasi-Steady Freezing of Soils

Yoshisuke Nakano

March 1994



Abstract

The results of mathematical and experimental studies presented in preceding reports clearly show that the model M_1 accurately describes the properties of a frozen fringe when the steady growth of an ice layer occurs. In this work the steady growth of ice-rich frozen soil is studied by using M_1. Deriving a traveling wave solution to the problem, we have found that the condition of steady growth of ice-rich frozen soil is uniquely determined by a set of two physical variables, such as α_0 and α_1 used earlier, under given hydraulic conditions and overburden pressures and that the traveling wave solution converges to the solution to the problem of a steadily growing ice layer when the velocity of the 0°C isotherm relative to the unfrozen part of the soil vanishes.

Cover: Temperature gradients α_1 and α_0.

For conversion of SI metric units to U.S./British customary units of measurement consult *Standard Practice for Use of the International System of Units (SI)*, ASTM Standard E380-89a, published by the American Society for Testing and Materials, 1916 Race St., Philadelphia, Pa. 19103.

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US Army Corps
of Engineers
Cold Regions Research &
Engineering Laboratory

Traveling Wave Solutions to the Problem of Quasi-Steady Freezing of Soils

Yoshisuke Nakano

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Prepared for
OFFICE OF THE CHIEF OF ENGINEERS

PREFACE

This report was prepared by Dr. Yoshisuke Nakano, Chemical Engineer, of the Applied Research Branch, Experimental Engineering Division, U. S. Army Cold Regions Research and Engineering Laboratory. Funding was provided by DA Project 4A161102AT24, *Research in Snow, Ice and Frozen Ground*, Task SC, Work Unit F01, *Physical Processes in Frozen Soil*.

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NOMENCLATURE

a_0 function defined by eq 63g
a_1 function defined by eq 63h
b defined by eq 63c
b_0 function defined by eq 63d
b_1 function defined by eq 63e
b_2 function defined by eq 69b
b_3 function defined by eq 69c
B_i ith constituent of the mixture. Subscripts i = 1, 2, and 3 are used to denote unfrozen water, ice and soil minerals, respectively
c heat capacity of the mixture defined by eq 10c
c_0 defined by eq 41c
c_i heat capacity of the ith constituent
d unit of time, day
d_i density of the ith constituent
f_i mass flux of the ith constituent relative to that of soil minerals where $i = 1, 2$
f_{ij} mass flux of the ith constituent relative to that of soil minerals in R_j where $i = 1, 2$ and $j = 0, 1, 2$
h_i heat content of the ith constituent
I_0 function defined by eq 56a
I_1 function defined by eq 56b
I_2 function defined by eq 59b
k thermal conductivity of the mixture
k_i thermal conductivity in R_i where $i = 0, 1, 2$
k_{11} limiting value of k_1 defined by eq 40h
k_{21} limiting value of k_2 defined by eq 48
K_0 hydraulic conductivity in the unfrozen part of the soil
K_i empirical function defined by eq 1 where i = 1, 2
K_{i1} limiting value of K_i as ξ approaches n_1 while ξ is in R_1, $i = 1,2$
K_{i0} limiting value of K_i as ξ approaches n_0 while ξ is in R_1, $i = 1,2$
L latent heat of fusion of water, 334 J g^{-1}
m boundary where the content of unfrozen water is negligible
M_i name of a model defined in Part I where i = 1, 2, 3
n boundary in R_0 and also used a generic moving surface
$\dot{n}$ velocity of $n = dn/dt$
n_i boundary with $i = 0, 1$ where n_0 denotes the boundary where $T = 0°C$ and n_1 the interface between R_2 and a frozen fringe
p_o gravity term, 0.098 kPa/cm
P_i pressure of the ith constituent where $i = 1, 2$
P_{10} value of P_1 at n_0
P_{1n} value of P_1 at n
P_{21} value of P_2 at n_1
q heat flux in the mixture by conduction defined by eq 8b
q^+ limiting value of q as ξ approaches n_1 while ξ is in R_1
q^- limiting value of q as ξ approaches n_1 while ξ is in R_2
q_i heat flux in the ith constituent by conduction
r rate of frost heave
R_0 unfrozen part of the soil
R_1 frozen fringe
R_2 frozen part of the soil
R_m region in the diagram of temperature gradients where an ice layer melts
R_s region in the diagram of temperature gradients where the steady growth of an ice layer occurs
R_s^* boundary between R_s and R_u
R_u region in the diagram of temperature gradients where the steady growth of an ice layer does not occur
s defined by eq 32b
s^+ defined by eq 31b
S defined by eq 71e
S_i defined by eq 70d–70g where $i = 1, 2, \cdot, 4$
t time
T temperature of the mixture
T_i temperature at n_i where $i = 0, 1$
T_o temperature at n_0 and used also as a reference temperature
v_i velocity of the ith constituent where $i = 1, 2, 3$
v_{ij} v_i in R_j where $i = 1, 2, 3$ and $j = 0, 1, 2$

V defined by eq 20

V_j V in R_j where $j = 0, 1, 2$

w_i defined by eq 35b where $i = 0, 1, 2$

w_{ij} defined by eq 35a where $i = 0, 1$ and $j = 0, 1, 2$

x spatial coordinate

X defined by eq A1

Y defined by eq B4

Y_1 defined by eq B6

z defined by eq 10b

z_1 defined by eq 44c

α_0 absolute value of the temperature gradient at n_0

α_1 absolute value of the limiting temperature gradient as ξ approaches n_1 while ξ is in R_2, defined by eq 47

β_0 defined by eq 41b

β_1 defined by eq 46

γ constant, 1.12 MPa °C^{-1}

δ thickness of a frozen fringe

δ_0 defined by eq 54c

ε defined by eq 58b

η defined by eq 40f

θ_i volumetric content of the ith constituent

λ_i rate of supply of mass of the ith constituent per unit volume of the mixture

$\bar{\lambda}_i$ rate of surface supply of mass of the ith constituent per unit surface of the mixture defined by eq 14a

Λ function defined by eq 26f

Λ_1^+ function defined by eq 23f

μ defined by eq 24e

μ_0 defined by eq 34a

ν defined by eq 22h

$\bar{\nu}$ defined by eq B8

ν_1 defined by eq 23i

$\bar{\nu}_1$ $\bar{\nu}$ at $T = T_1$

ξ coordinate defined by eq 16

ξ frost heave ratio defined by eq 72

π_0 defined by eq B2

π_1 defined by eq B3

ρ_i bulk density of the ith constituent

ρ_{ij} ρ_i in R_j

σ effective stress defined by eq 60b

σ_1 defined by eq 59a

ϕ_0 empirical function of T defined by eq 55a

ϕ_{01} value of ϕ_0 at $T = T_1$

ϕ_1 empirical function of T defined by eq 55b

ϕ_{11} value of ϕ_1 at $T = T_1$

ϕ_2 empirical function of T defined by eq 55c

ϕ_{21} value of ϕ_2 at $T = T_1$

ψ some function of x and t

$|\psi|$ jump of ψ defined by eq 11b

ψ^+ defined by eq 12b

ψ^- defined by eq 12a

$*$ superscript used to indicate the value of any variable evaluated when a point (α_1, α_0) in the diagram of temperature gradients is on R_s^*

Traveling Wave Solutions to the Problem of Quasi-Steady Freezing of Soils

YOSHISUKE NAKANO

INTRODUCTION

We will consider the one-directional steady growth of an ice layer. Let the freezing process advance from the top down and the coordinate x be positive upward with its origin fixed at some point in the unfrozen part of the soil. A freezing soil in this problem may be considered to consist of three parts: the unfrozen part R_0, the frozen fringe R_1 and the ice layer R_2 as shown in Figure 1. The physical properties of parts R_0 and R_2 are well understood, but our knowledge of the physical properties and the dynamic behavior of part R_1 does not appear sufficient for engineering applications.

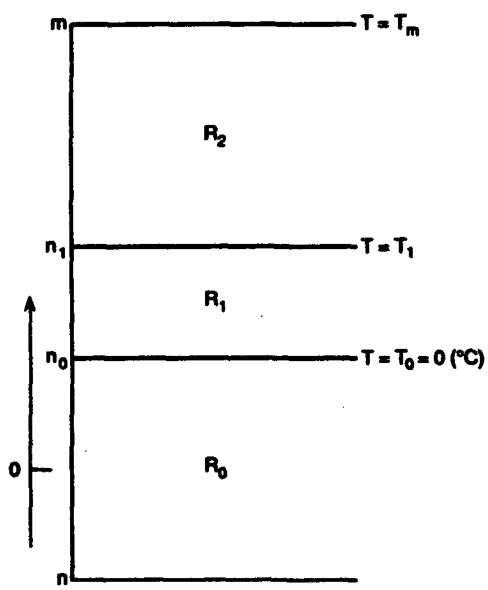

Figure 1. Steady growth of ice-rich frozen soil.

The results of our mathematical and experimental study on the steady growth condition of an ice layer were presented in the three previous reports (Nakano 1990, Takeda and Nakano 1990, Nakano and Takeda 1991). These results clearly show that the model M_1 accurately describes the properties of a frozen fringe during the steady growth of an ice layer under negligible overburden pressure. The model M_1 is the frozen fringe where ice may exist but does not grow during the steady growth of an ice layer and the mass flux of water f_1 is given as

$$f_1 = -K_1 \frac{\partial P_1}{\partial x} - K_2 \frac{\partial T}{\partial x} \qquad (1)$$

where x is the space coordinate, and K_1 and K_2 are the properties of a given soil that generally depend on temperature T and the composition of the soil.

Nakano (1990) has shown that the velocity $\dot{n}_0 (= d n_0 / dt)$ of the frost front is nonpositive and vanishes when the steady growth of an ice layer occurs. In this work we will study a case in which $\dot{n}_0$ is a given negative constant and the steady growth of ice-rich frozen soil, instead of an ice layer, takes place. In such a case the frozen fringe R_1 also moves downward with a constant speed. Ice may or may not exist in R_1. However, if a certain steady distribution of ice is present in R_1, then the growth of ice must occur in R_1 because unlike the case of a steadily growing ice layer, the velocity of soil particles in R_1 relative to $\dot{n}_0$ does not vanish when $\dot{n}_0 < 0$. Because of this we must modify the definition of M_1 so that ice may grow in R_1 when $\dot{n}_0 < 0$.

The objective of this work is to show that there exists a traveling wave solution to the problem of steadily growing ice-rich frozen soil and that this solution is reduced to the solution to the problem of a steadily growing ice layer obtained in the previous report (Nakano 1990) when the velocity n_0 vanishes. We will also show that the condition of a the steady growth of ice-rich frozen soils under given hydraulic conditions and applied pressures is uniquely determined by a set of two physical variables, such as α_0 and α_1, used in the previous reports.

BASIC EQUATIONS

We will treat the soil as a mixture of water in the liquid phase B_1, ice B_2 and soil minerals B_3 with bulk densities ρ_1, ρ_2 and ρ_3, respectively. If d_i is the density of the ith constituent, then the volumetric content θ_i of the ith constituent is given as

$$\theta_i = \rho_i / d_i . \qquad (2)$$

It is clear that the sum of θ_i should be unity, namely

$$\theta_1 + \theta_2 + \theta_3 = 1. \qquad (3)$$

We will assume that the density of each constituent of the mixture remains constant. Thus, the results of this study are accurate if the deformation of each constituent is negligibly small, regardless of overburden pressure. The dry density of the unfrozen part of the soil is assumed constant during the freezing process.

We will assume that the unfrozen part of the soil is kept saturated with water at all times by using an appropriate device of water supply. The balance of mass for the ith constituent is given (Nakano 1986) as

$$\frac{\partial}{\partial t} \rho_i = -\frac{\partial}{\partial x}(\rho_i v_i) + \lambda_i, \qquad i = 1,2,3 \qquad (4)$$

where v_i is the velocity of the ith constituent and λ_i is the time rate of supply of mass of the ith constituent per unit volume of the mixture. It should be mentioned that the summation convention on index i is not in force here, so that $(\rho_i v_i)$ represents only one term. Since none of the constituent is involved in chemical reaction, we have

$$\lambda_1 + \lambda_2 = 0 \quad \text{and} \quad \lambda_3 = 0. \qquad (5)$$

The balance of heat for the ith constituent is given as (Nakano 1986)

$$\frac{\partial}{\partial t}(\rho_i h_i) = -\frac{\partial}{\partial x}(\rho_i h_i v_i) - \frac{\partial}{\partial x} q_i, \qquad i = 1,2,3 \quad (6)$$

where $\rho_i h_i$ is the heat content of the ith constituent per unit bulk volume and q_i the heat flux by conduction. We assume that the constituents are locally in thermal equilibrium with each other, i.e., that the constituents have locally a common temperature T (°C). Under such an assumption, the heat content h_i is given as

$$h_1 = c_1(T - T_0) \qquad (7a)$$

$$h_2 = -L + c_2(T - T_0) \qquad (7b)$$

$$h_3 = c_3(T - T_0). \qquad (7c)$$

We sum up eq 6 for $i = 1, 2, 3$ to obtain the heat balance equation of the mixture given as

$$\frac{\partial}{\partial t} \sum_i \rho_i h_i = -\frac{\partial}{\partial x} \sum_i \rho_i h_i v_i - \frac{\partial}{\partial x} q \qquad (8a)$$

where q is defined as

$$q = \sum_i q_i. \qquad (8b)$$

It is known that q_i depends on the bulk density ρ_i, the thermal properties of ith constituent, the temperature gradient and the way in which the ith constituent is distributed in the mixture. For the sake of simplicity we will approximate q as

$$q = -k \frac{\partial T}{\partial x} \qquad (8c)$$

where k is the thermal conductivity of the mixture that generally depends on the thermal properties of each constituent and the composition of the mixture.

Using eq 4, we reduce eq 8a to

$$\sum_i \rho_i \frac{\partial}{\partial t} h_i = -\sum_i h_i \lambda_i - \sum_i \rho_i v_i \frac{\partial}{\partial x} h_i - \frac{\partial}{\partial x} q . \qquad (9)$$

We will assume that c_i and L in eq 7a, 7b and 7c do not depend on T. Choosing T_0 to be 0 (°C) and using eq 7a, 7b and 7c, we reduce eq 9 to:

$$\frac{\partial}{\partial x} q = L(\lambda_2 + z) \qquad (10a)$$

where z is defined as

$$Lz = -c\frac{\partial T}{\partial t} + (c_1 - c_2)T\lambda_2 - \sum_i \rho_i v_i c_i \frac{\partial T}{\partial t} \quad (10b)$$

$$c = c_1\rho_1 + c_2\rho_2 + c_3\rho_3 . \quad (10c)$$

We will now consider a moving surface whose location is given as

$$x = n(t) \leq m(t). \quad (11a)$$

In a neighborhood of $n(t)$ we choose two moving surfaces $n^-(t)$ and $n^+(t)$ with $n^-(t) > n(t) > n^+(t)$. The jump of a quantity $\psi(x,t)$ at $n(t)$ is defined as

$$|\psi| = \psi^- - \psi^+ \quad (11b)$$

where

$$\psi^- = \lim_{n^- \to n} \psi \quad (12a)$$

$$\psi^+ = \lim_{n^+ \to n} \psi \quad (12b)$$

It is clear that $|\psi| = 0$ if ψ is continuous at $n(t)$.

Jump conditions at $n(t)$ under the assumption of a continuous T are given (Nakano 1986) as

$$|\rho_i v_i| = |\rho_i|\,\dot{n} + \overline{\lambda}_i \qquad i = 1, 2, 3 \quad (13a)$$

$$|q| = \left|\sum_i \rho_i h_i\right| \dot{n} - \left|\sum_i v_i \rho_i h_i\right| \quad (13b)$$

where $\overline{\lambda}_i$ is the surface supply of the mass of the ith constituent defined as

$$\overline{\lambda}_i = \lim_{n^+,\, n^- \to n} \int_{n^+}^{n^-} \lambda_i \, dx \quad (14a)$$

and $\dot{n}$ is assumed to be continuous and is defined as

$$\dot{n} = \frac{dn}{dt} \quad (14b)$$

From eq 5 we obtain

$$\overline{\lambda}_1 + \overline{\lambda}_2 = 0 \qquad \text{and} \qquad \overline{\lambda}_3 = 0. \quad (15)$$

The jump conditions, eq 13a and 13b, are necessary and sufficent conditions for the conservation law of heat and mass to hold at n. In other words, the conservation law of either heat or mass must be violated if one of these conditions does not hold true at n.

Quasi-steady problem

We will consider a special case in which a frost front $x = n_0(t)$ moves with a constant velocity $\dot{n}_0$. In such a case we will seek a quasi-steady solution to the problem described by eq 4 and 10a in the form of a traveling wave. We will introduce a new independent variable ξ defined as

$$\xi = x - \dot{n}_0 t \quad (16)$$

where $\dot{n}_0 = d\dot{n}_0(t)/dt$. Using eq 16 we reduce eq 4 to

$$\frac{d}{d\xi}\rho_i\left(v_i - \dot{n}_0\right) = \lambda_i\,, \qquad i = 1, 2, 3 \quad (17)$$

For the sake of convenience we will define new dependent variables f_1 and f_2 as

$$f_1 = \rho_1\left(v_1 - v_3\right) \quad (18a)$$

$$f_2 = \rho_2\left(v_2 - v_3\right) \quad (18b)$$

It is easy to see that f_i ($i = 1, 2$) is the mass flux of either B_1 or B_2 relative to the mass flux of soil particles. Using eq 18a and 18b, we reduce eq 17 to

$$(\rho_1 V)' = -f_1' - \lambda_2 \quad (19a)$$

$$(\rho_2 V)' = -f_2' + \lambda_2 \quad (19b)$$

$$(\rho_3 V)' = 0 \quad (19c)$$

where primes denote differentiation with respect to ξ and V is defined as

$$V = v_3 - \dot{n}_o . \quad (20)$$

Using eq 16, 18a and 18b, we will reduce eq 10a and 10b to

$$q' = -(kT')' = L(\lambda_2 + z) \quad (21a)$$

$$Lz = -(c_1 f_1 + c_2 f_2 + cV)T' + (c_1 - c_2)\lambda_2 T. \quad (21b)$$

Traveling wave solution

We will derive a traveling wave solution that satisfies the jump conditions, eq 13a and 13b, at n_1, and the balance equations of mass and heat, eq 19a 19b 19c and 21a. We will assume that the boundaries n, n_0, n_1 and m in Figure 1 move with the same

constant velocity, namely:

$$\dot{n} = \dot{n}_0 = \dot{n}_1 = \dot{m}. \tag{22a}$$

The pressure P_1 of water is assumed to be a given constant at n as

$$P_1(n) = P_{1_n}. \tag{22b}$$

According to M_1 (Nakano 1990), the mass flux of water f_{11} in R_1 is given as

$$f_{11} = -K_1 \frac{\partial P_1}{\partial \xi} - K_2 \frac{\partial T}{\partial \xi} \quad \text{for } \xi \text{ in } R_1 \tag{22c}$$

$$K_2 / K_1 \to \gamma \qquad \text{as} \qquad f_{11} \to 0 \tag{22d}$$

$$\lim_{\substack{\xi \to n_1 \\ \xi \text{ in } R_1}} P_1(\xi) = P_2(n_1) = P_{21} \tag{22e}$$

where P_2 is the pressure of the frozen part of the soil R_2 and γ is a constant with the value of 1.12 MPa °C^{-1}.

We will also assume that the composition of the soil is continuous and $\overline{\lambda}_2$ vanishes at n_0 as

$$|\rho_i| = 0, \qquad \overline{\lambda}_2 = 0 \qquad \text{at } n_0. \tag{22f}$$

The assumption of eq 22f implies that the velocity v_i and the flux of heat q are continuous at n_0. We will assume that the movement of ice relative to soil particles is negligibly small everywhere, that is

$$f_2 = 0 \qquad \text{in } R_i \ (i = 0,1,2). \tag{22g}$$

As we discussed (Nakano 1990), when the steady growth of an ice layer occurs, the pressure P_1 is continuous but the first derivative dP_1/dx of P_1 may be discontinuous at n_0. Therefore, the boundary n_0 is generally a free boundary where the first derivative $dP_1/d\xi$ may be discontinuous. Finally we will assume that ρ_1 is given as

$$\rho_1 = \rho_3 v(T) \tag{22h}$$

where v(T) is a given empirically determined function of T that is assumed to be approximated by the equilibrium unfrozen water content at T.

We will now denote the values of v_i and V, for instance, in the part $R_j (j = 0,1,2)$ by v_{ij} and V_j, respectively. We will denote the limiting value of V_1, for instance, as ξ approaches n_1 with $\xi < n_1$ by V_1^+ and that of V_1 as ξ approaches n_0 with $\xi > n_0$ by V_1^- as shown in Figure 2. Our immediate task is to reduce the number of unknown variables appearing in this figure by using the jump conditions and the balance equations of mass and heat.

Integrating eq 19a, 19b and 19c from $\xi = n_1^-$ to m^+, we obtain

$$f_{12}^+ + \rho_{12}^+ V_2^+ = f_{12}^- + \rho_{12}^- V_2^- - \Lambda_1^+ \tag{23a}$$

$$\rho_{22}^+ V_2^+ = \rho_{22}^- V_2^- + \Lambda_1^+ \tag{23b}$$

$$\rho_{32}^+ V_2^+ = \rho_{32}^- V_2^- \tag{23c}$$

where

$$V_2^+ = v_{32}^+ - \dot{n}_0 = r - \dot{n}_0 \tag{23d}$$

$$V_2^- = v_{32}^- - \dot{n}_0 \tag{23e}$$

$$\Lambda_1^+ = \int_{n_1^-}^{m^+} \lambda_2 d\xi \tag{23f}$$

where it is assumed that the temperature T_m at $\xi = m$ is low so that f_{12}^+ and ρ_{12}^+ vanish. This implies that v_{32}^+ is equal to the rate of frost heave r. Eliminating Λ_1^+, we will reduce eq 23a, 23b and 23c to

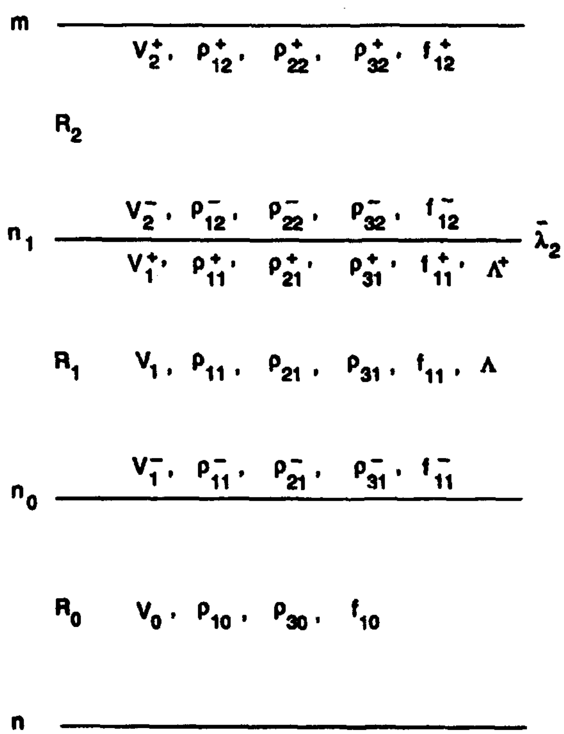

Figure 2. Variables in R_0, R_1 and R_2.

$$\rho_{32}^{+}V_2^{+} = \left(\nu_1\rho_{32}^{-} + \rho_{22}^{-}\right)V_2^{-} + f_{12}^{-} \tag{23g}$$

$$\rho_{32}^{+}V_2^{+} = \rho_{32}^{-}V_2^{-} \tag{23h}$$

where ν_1 is defined as:

$$\nu_1 = \nu(T_1). \tag{23i}$$

Using eq 3, we obtain

$$\rho_{22}^{+} = d_2\left(1 - d_3^{-1}\,\rho_{32}^{+}\right) \tag{23j}$$

$$\rho_{22}^{-} = d_2\left[1 - \left(\nu_1 d_1^{-1} + d_3^{-1}\right)\rho_{32}^{-}\right]. \tag{23k}$$

From eq 23f, 23i and 23j, we obtain

$$V_2^{+} = \left[1 + \nu_1\left(d_2^{-1} - d_1^{-1}\right)\rho_{32}^{-}\right] V_2^{-} + d_2^{-1} f_{12}^{-}. \tag{23ℓ}$$

It is easy to see that all variables at m^{+} are determined if all variables at n_1^{-} are known. Using eq 13a at n_1, we obtain

$$f_{11}^{+} + \rho_{11}^{+}V_1^{+} = f_{12}^{-} + \rho_{12}^{-}V_2^{-} + \bar{\lambda}_2 \tag{24a}$$

$$\rho_{21}^{+}V_1^{+} = \rho_{22}^{-}V_2^{-} - \bar{\lambda}_2 \tag{24b}$$

$$\rho_{31}^{+}V_1^{+} = \rho_{32}^{-}V_2^{-} \tag{24c}$$

where

$$V_1^{+} = v_{31}^{+} - \dot{n}_0. \tag{24d}$$

For the sake of simplicity, we will introduce a new variable μ defined as

$$\rho_{32}^{-} = \mu\,\rho_{31}^{+}. \tag{24e}$$

Using eq 22h and 24e, we will reduce eq 24a, 24b and 24c to

$$f_{11}^{+} + \nu_1\rho_{31}^{+}V_1^{+} = f_{12}^{-} + \nu_1\mu\,\rho_{31}^{+}V_2^{-} + \bar{\lambda}_2 \tag{25a}$$

$$\rho_{21}^{+}V_1^{+} = \rho_{22}^{-}V_2^{-} - \bar{\lambda}_2 \tag{25b}$$

$$V_1^{+} = \mu V_2^{-}. \tag{25c}$$

Using eq 3, we obtain

$$\nu_1\mu\rho_{31}^{+}d_1^{-1} + \rho_{22}^{-}d_2^{-1} + \mu\rho_{31}^{+}d_3^{-1} = 1 \tag{25d}$$

$$\nu_1\,\rho_{31}^{+}d_1^{-1} + \rho_{21}^{+}d_2^{-1} + \rho_{31}^{+}d_3^{-1} = 1 \tag{25e}$$

From eq 13b, we obtain

$$|q| = \bar{\lambda}_2[L + (c_1 - c_2)\,T_1]. \tag{25f}$$

Integrating eq 19a 19b and 19c from $\xi = n_0$ to ξ, we obtain the following equations given as

$$f_{11} + \nu\rho_{31}V_1 = f_{10} + \rho_{10}V_1^{-} - \Lambda \tag{26a}$$

$$\rho_{21}V_1 = \Lambda \tag{26b}$$

$$\rho_{31}V_1 = \rho_{30}V_1^{-} \tag{26c}$$

where

$$V_1 = v_{31} - \dot{n}_0 \tag{26d}$$

$$V_1^{-} = V_0 = -\,\dot{n}_0 \tag{26e}$$

$$\Lambda(\xi) = \int_{n_0}^{\xi} \lambda_2 d\xi, \qquad \xi \ge n_0. \tag{26f}$$

Using eq 3, we obtain

$$\nu\rho_{31}\,d_1^{-1} + \rho_{21}\,d_2^{-1} + \rho_{31}\,d_3^{-1} = 1 \tag{26g}$$

$$\rho_{10}\,d_1^{-1} + \rho_{30}\,d_3^{-1} = 1. \tag{26h}$$

Taking limits of eq 26a, 26b and 26c as ξ approaches n_1, we obtain

$$f_{11}^{+} + \nu_1\rho_{31}^{+}V_1^{+} + \Lambda^{+} = f_{10} + \rho_{10}V_0 \tag{27a}$$

$$\rho_{21}^{+}V_1^{+} - \Lambda^{+} = 0 \tag{27b}$$

$$\rho_{31}^{+}V_1^{+} = \rho_{30}V_0. \tag{27c}$$

Comparing eq 25a, 25b and 25c with eq 27a, 27b and 27c, respectively, we obtain

$$f_{12}^{-} + \nu_1\mu\rho_{31}^{+}V_2^{-} = f_{10} + \rho_{10}V_0 - \Lambda^{+} - \bar{\lambda}_2 \tag{28a}$$

$$\rho_{22}^{-}V_2^{-} = \Lambda^{+} + \bar{\lambda}_2 \tag{28b}$$

$$\mu V_2^{-} = \left(\rho_{30}/\rho_{31}^{+}\right)V_0. \tag{28c}$$

Let us assume for the time being that $\dot{n}_0$, ν_1 (or T_1) and $\bar{\lambda}_2$ are given. It is easy to see that the left-hand terms of eq 28a, 28b and 28c contain four unknown variables at n_1^{-} in R_2; V_2^{-}, ρ_{22}^{-}, μ, and f_{12}^{-}, while the right-hand terms of these three equations contain unknown variables in the combined region of $R_0 + R_1$. Since ρ_{22}^{-} and μ are related by eq 25d, all the unknown variables at n_1^{-} listed in Figure 2 are uniquely determined if all the variables in $R_0 + R_1$ listed in Figure 2 are known.

From eq 27a, 27b and 27c we find that these three equations contain five unknown limiting values, V_1^+, ρ_{21}^+, ρ_{31}^+, f_{11}^+ and Λ^+. Since ρ_{21}^+ and ρ_{31}^+ are related by eq 25e, we have actually four unknown limiting values and three equations. Therefore, if one of these four unknowns is given, then all six unknown limiting values at n_1^+ listed in Figure 2 are uniquely determined. Choosing V_1 to be an independent variable, we will write all other limiting values as follows. First from eq 27c, we obtain:

$$\rho_{31}^+ = \rho_{30}(V_0/V_1^+) \tag{29a}$$

$$\rho_{11}^+ = \nu_1 \rho_{30}(V_0/V_1^+) . \tag{29b}$$

From eq 25e, we obtain:

$$\rho_{21}^+ = d_2 - d_2(\nu_1 d_1^{-1} + d_3^{-1})\rho_{30}(V_0/V_1^+) . \tag{29c}$$

From eq 27a and 27b, we obtain

$$f_{11}^+ = f_{10} + \rho_{10}V_0 - (\nu_1\rho_{31}^+ + \rho_{21}^+)V_1^+ \tag{30a}$$

$$\Lambda^+ = \rho_{21}^+ V_1^+. \tag{30b}$$

Substituting ρ_{21}^+ in eq 30a by eq 29c, we will reduce eq 30a to

$$f_{11}^+ = f_{10} + s^+ V_0 - d_2(V_1^+ - V_0) \tag{31a}$$

where s^+ is defined as

$$s^+ = (1 - d_1^{-1}d_2)(\rho_{10} - \nu_1\rho_{30}) . \tag{31b}$$

Unknown variables, ρ_{31}, ρ_{11}, ρ_{21}, f_{11} and Λ are given by eq 29a, 29b, 29c, 31a and 30b, respectively, in which superscripts + are deleted and ν_1 is replaced by ν. For instance, f_{11} is given as

$$f_{11} = f_{10} + sV_0 - d_2(V_1 - V_0) \tag{32a}$$

where s is given as

$$s = (1 - d_1^{-1}d_2)(\rho_{10} - \nu\rho_{30}) . \tag{32b}$$

Using eq 28a and 28c, we will write unknown variables at n_1^- as

$$V_2^- = d_2^{-1}\bar{\lambda}_2 + V_1^+ \tag{33a}$$

$$\mu = V_1^+(V_1^+ + d_2^{-1}\bar{\lambda}_2)^{-1} \tag{33b}$$

$$\rho_{32}^- = \mu\rho_{31}^+ \tag{33c}$$

$$\rho_{12}^- = \nu_1\mu\rho_{31}^+ \tag{33d}$$

$$\rho_{22}^- = d_1[1 - (\nu_1 d_1^{-1} + d_3^{-1})\mu\rho_{31}^+] \tag{33e}$$

$$f_{12}^- = f_{11}^+ - \bar{\lambda}_2 . \tag{33f}$$

Using eq 31a, we will reduce eq 33f to

$$f_{12}^- = f_{10} + s^+ V_0 - d_2(V_1^+ - V_0) - \bar{\lambda}_2 . \tag{33g}$$

In actual experiments, ρ_{10} and ρ_{30} are given as initial conditions. If V_0, ν_1 (or T_1), λ_2, f_{10} and V_1 are given, then all other variables are uniquely determined. Since ρ_{31}^+ is difficult to measure experimentally, it is convenient to introduce a new variable μ_0 defined as

$$\mu_0 = \rho_{32}^+\rho_{30}^{-1} = (\rho_{32}^+/\rho_{32}^-)(\rho_{31}^+/\rho_{30})\mu. \tag{34a}$$

Using eq 23h, 25c and 27c, we will reduce eq 34a to

$$\mu_0 = V_0/V_2^+ . \tag{34b}$$

Using eq 24e and 25c, we will reduce eq 23ℓ to

$$V_2^+ = \mu^{-1}[1 + \nu_1(d_1^{-1} - d_2^{-1})\mu\rho_{31}^+]V_1^+ + d_2^{-1}f_{12}^- . \tag{34c}$$

Substituting f_{12}^- in eq 34c by eq 33g and using eq 33b, we will reduce eq 34c to

$$V_2^+ = d_2^{-1}f_{10} + V_0[1 + (d_2^{-1} - d_1^{-1})\rho_{10}] . \tag{34d}$$

Combining eq 34b and 34d, we obtain

$$\mu_0 = V_0\{d_2^{-1}f_{10} + V_0[1 + (d_2^{-1} - d_1^{-1})]\rho_{10}\}^{-1} . \tag{34e}$$

From eq 23d and 34d we obtain

$$r = d_2^{-1}f_{10} + (d_2^{-1} - d_1^{-1})\rho_{10}V_0 \tag{34f}$$

Using eq 34f, we will reduce eq 34e to

$$\mu_0 = V_0(r + V_0)^{-1} . \tag{34g}$$

We will introduce new variables, w_{ij} and w_j, defined as

$$w_{ij} = \rho_{ij}\rho_{3j}^{-1}, \quad i = 1,2 \text{ and } j = 0, 1, 2 \tag{35a}$$

$$w_j = w_{1j} + w_{2j} \quad j = 0, 1, 2. \tag{35b}$$

It is clear that w_{ij} is the content of the ith constituent in R_j and that w_j is the content of ice and unfrozen water in R_j. We will refer to w_j as the total water

content. Using w_1, we will reduce eq 32a to

$$f_{11} = f_{10} + \rho_{30}(w_0 - w_1)V_0 . \tag{36a}$$

Using eq 26a, 26b and 36a, we obtain

$$\Lambda = \rho_{30}(w_1 - \nu)V_0 . \tag{36b}$$

We will now examine the behavior of f_{11}. From eq 32a we obtain

$$f_{11} = f_{10} + sV_0 - d_2 v_{31} . \tag{37a}$$

Using eq 26c, we will write v_{31} as:

$$v_{31} = \left(\rho_{30}\,\bar{\rho_{31}} - 1\right) V_0 . \tag{37b}$$

Since T decreases as ξ increases from n_0 to n_1, s is positive in R_1 and increases with ξ. It is anticipated that ρ_{31} may decrease with ξ but does not increase with ξ. Since $V_0 \geq 0$, from eq 37b we find that $v_{31} \geq 0$ and $v'_{31} \geq 0$ in R_1. Therefore, from eq 37a we obtain

$$f_{10} + s^+ V_0 \geq f_{11} \geq f_{10} - d_2 v_{31}^+ \tag{38a}$$

where

$$v_{31}^+ = \left(\rho_{30} - \rho_{31}^+\right) V_0 / \rho_{31}^+ . \tag{38b}$$

Ice-rich frozen soil

We will focus the remainder of our analysis on a special case in which the frozen part of the soil contains a significant amount of ice. For such a case the mobility of water in R_2 is anticipated to be much less than that in R_1 and we may neglect f_{12}^-. It follows from eq 34a, 34g and 36b that the values of μ_0, V_0 and Λ remain small.

The exact composition of R_1 is not known. However, it is a generally accepted view that ρ_{31} does not change significantly from ρ_{30}. The results of our analysis on the data of Tomakomai silt (Nakano and Takeda 1991) appear to support such a viewpoint. Assuming the existence of a certain rule for ρ_{31}, we will explore probable rules below. Suppose that such a rule is known, then two of five independent variables, V_1 and λ_2, are uniquely determined by eq 26c and 33g, respectively, when three remaining independent variables, V_0, T_1 and f_{10}, are given in this case. Let us consider first a special rule that ρ_{31} is kept constant at ρ_{30}. In such a case v_{31} vanishes and eq 37a is reduced to

$$f_{11} = f_{10} + sV_0 . \tag{39a}$$

When $V_0 > 0$, we obtain

$$f_{11} > f_{10} \text{ and } f'_{11} > 0, \quad \textit{if } V_0 > 0 \text{ and } \xi > n_0 \tag{39b}$$

It follows from eq 39b that f_{11} is greater than f_{10} and increases with ξ. This special case does not appear to be probable because the mobility of water should not increase with increasing ξ.

Next we will consider a special rule that the total water content w_1 is kept constant at w_0. From eq 36a and 33f we obtain:

$$f_{11} = f_{10} = \bar{\lambda}_2 . \tag{40a}$$

From eq 37a we obtain

$$v_{31} = d_2^{-1}\, sV_0 . \tag{40b}$$

From eq 29a we obtain

$$\rho_{31} = \rho_{30}\left(1 + d_2^{-1}\, s\right)^{-1} . \tag{40c}$$

We will reduce eq 36b to

$$\Lambda = \rho_{30}(w_0 - \nu)V_0 . \tag{40d}$$

It follows from eq 40b and 40c that v_{31} increases with ξ while ρ_{31} decreases with ξ for a given V_0. This second special case appears more probable than the first case because the mass flux of water should not increase with increasing ξ. We will study the second case below. The empirical function $\nu(T)$ in eq 40d is known to be an increasing function of T with $\nu(0) = w_0$. We will assume that $\nu(T)$ possesses a continuous first derivative.

The thermal conductivity k_1 of R_1 depends on the composition of R_1. Our experimental data indicate that k_1 is a nondecreasing function of ξ. We will approximate k_1 by a linear function of ξ as

$$k_1(\xi) = k_0[1 + \eta(\xi - n_0)], \quad n_1 > \xi \geq n_0 \tag{40e}$$

$$\eta = (k_{11} - k_0)/(\delta k_0) \geq 0 \tag{40f}$$

$$\delta = n_1 - n_0 \tag{40g}$$

$$\lim_{\substack{\xi \to n_1 \\ \xi \text{ in } R_1}} k_1(\xi) = k_{11} \leq k_{21} \tag{40h}$$

where k_{21} is the limiting value of k_2 when ξ approaches n_1 while ξ is in R_2. Under assumptions described above we will study thermal and hydraulic fields below.

Temperature $T(\xi)$

We will seek solutions $T(\xi)$ to the balance equation of heat given by eq 21a in R_0 and R_1. We will begin with R_0. Since $f_1 = f_{10}, f_2 = 0$, $V = V_0$ and $\lambda_2 = 0$, from eq 21a and 21b we obtain

$$T'' - \beta_0 T' = 0, \qquad \xi \text{ in } R_0 \tag{41a}$$

$$\beta_0 = (c_1 f_{10} + c_0 V_0)/k_0 \tag{41b}$$

where k_0 is the thermal conductivity of R_0 and c_0 is defined as

$$c_0 = c_1\rho_{10} + c_3\rho_{30}\,. \tag{41c}$$

Since $T(n_0) = 0°C$, integrating eq 41a, we obtain

$$T(\xi) = \alpha_0\beta_0^{-1}\{1 - \exp[-\beta_0(n_0 - \xi)]\} \tag{42a}$$

$$T'(\xi) = -\alpha_0 \exp[-\beta_0(n_0 - \xi)] \tag{42b}$$

where α_0 is defined as

$$a_0 = -T'(n_0). \tag{42c}$$

Next we will seek a solution in R_1. First we will rewrite eq 21a and 21b by using eq 26a, 26b and 26c. Since $f_1 = f_{10}, f_2 = 0$ and $\lambda_2 = \Lambda'$ in this case, we will reduce eq 21b to

$$Lz = -(c_1 f_{10} + cV_1)T' + (c_1 - c_2)\Lambda' T. \tag{43a}$$

Using eq 26a, 26b and 26c, we obtain

$$cV_1 = -(c_1 - c_2)\Lambda + c_0 V_0\,. \tag{43b}$$

Using eq 43b, we will reduce eq 43a to

$$Lz = -(c_1 f_{10} + c_0 V_0)T' + (c_1 - c_2)(\Lambda T)'. \tag{43c}$$

Using eq 43c, we will reduce eq 21a to

$$q' = -k_0\beta_0 T' + (c_1 - c_2)(\Lambda T)' + L\Lambda'. \tag{44a}$$

Integrating eq 44a from $\xi = n_0$ to ξ, we obtain

$$q = -k_1 T' = k_0\alpha_0 - z_1 T + L\Lambda \tag{44b}$$

$$z_1 = k_0\beta_0 - (c_1 - c_2)\Lambda \tag{44c}$$

where k_1 is the thermal conductivity of R_1 and Λ is given as

$$\Lambda = (w_0 - v)\,\rho_{30} V_0\,. \tag{44d}$$

Using eq 41b and 44d, we will reduce eq 44c to

$$z_1 = c_1 f_{10} + \rho_{30} V_0\,[c_3 + c_2 w_0 + (c_1 - c_2)v]. \tag{44e}$$

Neglecting the last term in eq 44e, we will reduce eq 44b to

$$-k_1 T' = k_0\alpha_0 - k_0\beta_1 T + L\Lambda. \tag{45}$$

where β_1 is defined as

$$\beta_1 = k_0^{-1}[c_1 f_{10} + \rho_{30} V_0(c_3 + c_2 w_0)]. \tag{46}$$

We will reduce the jump condition (eq 25f) to a somewhat more convenient form by using eq 44b. We will write the limiting value q^- when ξ approaches n_1 while ξ is in R_2 as

$$q^- = k_{21}\alpha_1 \tag{47}$$

$$k_{21} = \lim_{\substack{\xi \to n_1 \\ \xi \text{ in } R_2}} k_2(\xi) \tag{48}$$

where k_2 is the thermal conductivity of R_2 and α_1 is the absolute value of the limiting temperature gradient as ξ approaches n_1 while ξ is in R_2. Using eq 44b and 47, we will reduce eq 25f to

$$k_{21}\alpha_1 - k_0\alpha_0 + k_0\beta_0 T_1 = (f_{10} + \Lambda^+)\,[L + (c_1 - c_2)T_1] \tag{49}$$

$$\Lambda^+ = (w_0 - v_1)\rho_{30} V_0\,. \tag{50}$$

As shown in Appendix A, eq 45 has a unique and decreasing solution for $n_1 \ge \xi \ge n_0$. For a special case in which the following condition holds true,

$$\eta\delta < 1 \quad \text{and} \quad \beta_1\delta < 1. \tag{51}$$

We found that the condition, eq 51, holds true when the steady growth of an ice layer occurs (Takeda and Nakano 1990). When eq 51 holds true, we may use an approximation (Nakano 1990) given as

$$X^{\beta_1\eta^{-1}} = 1 + \beta_1(\xi - n_0). \tag{52}$$

We obtain an approximate solution (App. B) given as

$$T(\xi) = \pi_0\pi_1^{-1}\bar{v} - (\pi_1 - \pi_0\pi_1^{-1}\bar{v}\,\beta_1)(\xi - n_0) \tag{53a}$$

$$T'(\xi)X = -\pi_1 + \pi_0\left(\nu + \pi_1^{-1}\beta_1\bar{\nu}\right)$$
$$- \beta_1\left(\pi_1 - \pi_0\pi_1^{-1}\beta_1\bar{\nu}\right)(\xi - n_0). \quad (53b)$$

It is easy to see from eq 45 that the temperature $T(\xi)$ is uniquely determined if V_0, f_{10}, α_0 and ξ are given. Hence, the temperature T_1 at n_1 is determined by V_0, f_{10}, α_0 and δ. Suppose that V_0, f_{10}, α_0 and δ are given, then T_1 is determined by eq 45. Once T_1 is known, then α_1 is determined by eq 49. As we described in the preceding section, all variables listed in Figure 2 are determined by V_0, T_1 and f_{10} if ρ_{31} obeys a certain known rule in R_1. This implies that four independent variables must be given in order to specify the condition of freezing. We will choose α_0, α_1, f_{10} and V_0 to be independent variables.

Pressure $P_1(\xi)$

When the mass flux of water is given by eq 1, we have found (Nakano 1990) that the following equations hold true in R_1:

$$P_{10} = P_{1_n} - [(f_{10}/K_0) + p_0]\delta_0 \quad (54a)$$

$$P_{21} = P_{10} - \int_{n_0}^{n_1} K_1^{-1} K_2 T' \, d\xi - f_{10}\int_{n_0}^{n_1} K_1^{-1} \, d\xi \quad (54b)$$

where $P_{10} = P_1(n_0)$, $P_{1_n} = P_1(n)$,
n = some point in R_0
K_0 = hydraulic conductivity in R_0
p_0 = gravity term that is equal to the density d_1 multiplied by the gravitational acceleration.

δ_0 is defined as

$$\delta_0 = n_0 - n > 0. \quad (54c)$$

We will assume that P_{21}, P_{1_n} and δ_0 are given.

In order to reduce eq 54b to a simpler form, we will introduce the following three dimensionless quantities

$$\phi_0(T) = \begin{cases} 1 & T_1 = 0 \\ T_1^{-1}\int_0^T (K_{10}/K_1)(K_2/K_{20})\, dT & T < 0 \end{cases} \quad (55a)$$

$$\phi_1(T) = \begin{cases} 1 & T_1 = 0 \\ T_1^{-1}\int_0^T (K_{10}/K_1)(k_1/k_0)\, dT & T < 0 \end{cases} \quad (55b)$$

$$\phi_2(T) = \begin{cases} 1 & T_1 = 0 \\ T_1^{-1}\int_0^T (\nu/w_0)(K_{10}/K_1)(k_1/k_0)\, dT & T < 0 \end{cases} \quad (55c)$$

where K_{10} and K_{20} are the limiting values of K_1 and K_2, respectively, as ξ approaches n_0 while ξ in R_1.

Choosing T as an independent variable, we will write the two integrations in eq 54b as

$$I_0 \equiv \int_{n_0}^{n_1} K_1^{-1} K_2 T' d\xi = K_{10}^{-1} K_{20} T_1 \phi_{01} \quad (56a)$$

$$I_1 \equiv \int_{n_0}^{n_1} K_1^{-1} d\xi = \int_0^{T_1} (K_1 T')^{-1} dT \quad (56b)$$

where $\phi_{01} = \phi_0(T_1)$. We will write eq 54b as

$$P_{21} = P_{10} - I_0 - f_{10} I_1. \quad (57)$$

Using eq 55b and 55c, we will reduce eq 56b to (App. C)

$$I_1 = -(\pi_1 K_{10})^{-1}\left[\phi_{11} T_1 (1-\varepsilon) + w_0 \pi_0 \pi_1^{-1} \phi_{21} T_1\right] \quad (58a)$$

$$\varepsilon = \beta_1 \pi_1^{-1}\left[-T_1 + (\phi_{11})^{-1}\int_0^{T_1} \phi_1 \, dT\right] \quad (58b)$$

where $\phi_{11} = \phi_1(T_1)$ and $\phi_{21} = \phi_2(T_1)$.

Using eq 56a and 58a, we will write eq 57 as

$$\sigma_1 = P_{21} - P_{10} = -T_1 I_2 \quad (59a)$$

$$I_2 = K_{10}^{-1} K_{20} \phi_{01} - (\pi_1 K_{10})^{-1}$$
$$f_{10}\left[\phi_{11}(1-\varepsilon) + w_0 \pi_0 \pi_1^{-1} \phi_{21}\right]. \quad (59b)$$

Using eq 54a, we obtain

$$\sigma_1 = \sigma + p_0\delta_0 + \delta_0 K_0^{-1} f_{10} \quad (60a)$$

$$\sigma = P_{21} - P_{1_n} \quad (60b)$$

$$-T_1 = (\sigma + p_0\delta_0 + \delta_0 K_0^{-1} f_{10})/I_2. \tag{60c}$$

Since the composition of the freezing soil is assumed to be continuous at n_0, we may expect that the limiting value K_{10} and K_0 should be equal

$$K_0 = K_{10}. \tag{61a}$$

Neglecting the gravity term, we will reduce eq 60c to

$$-T_1 = (\sigma + \delta_0 K_0^{-1} f_{10})/I_2. \tag{61b}$$

When f_{10} vanishes, from eq 61b we obtain

$$\sigma = -(K_{20}/K_0)\,\phi_{01} T_1, \quad \text{if } f_{10} = 0. \tag{61c}$$

The generalized Clausius-Clapeyron equation (Edlefsen and Anderson 1943), which was proven empirically by Radd and Oertle (1973), is given as

$$\sigma = -\gamma T_1, \qquad \text{if } f_{10} = 0. \tag{62a}$$

Comparing eq 61c with eq 62a, we obtain

$$\gamma = (K_{20}/K_0)\phi_{01}, \quad \text{if } f_{10} = 0. \tag{62b}$$

It follows from eq 22d that eq 62b holds true and that we have

$$\gamma = K_{20}/K_0 \tag{62c}$$

$$\phi_{01} = 1, \qquad \text{if } f_{10} = 0. \tag{62d}$$

It should be noted that eq 62c should hold true regardless of f_{10}. Using eq 62c, we reduce eq 59b to

$$I_2 = \gamma\phi_{01} - (\pi_1 K_0)^{-1}$$
$$f_{10}\,[\phi_{11}(1-\varepsilon) + w_0\pi_0\pi_1^{-1}\phi_{21}]. \tag{62e}$$

For a special case in which σ is negligibly small, eq 61b is reduced to

$$-T_1 = (\delta_0 K_0^{-1} f_{10})/I_2. \tag{62f}$$

At the end of the preceding section we had four independent variables, α_0, α_1, f_{10} and V_0. Since we have derived another equation, eq 60c, we now have three independent variables, α_0, α_1 and V_0. We will derive one more equation below in order to reduce the number of independent variables to two.

Condition of steady growth

Since the mass flux is given by eq 22c, the flux f_{10} in a neighborhood of n_1 is given as

$$f_{10} = -K_{11} P_1'(n_1^+) - K_{21} T'(n_1^+) \tag{63a}$$

where K_{11}, $P_1'(n_1^+)$, K_{21} and $T'(n_1^+)$ are the limiting values of K_1, $P_1'(\xi)$, K_2 and $T'(\xi)$, respectively, as ξ approaches n_1 while ξ is in R_1. From eq 53b we obtain

$$T'(n_1^+) = -\alpha_0 b \tag{63b}$$

$$b = (1+\eta\delta)^{-1}(b_0 + b_1\delta) \tag{63c}$$

$$b = \alpha_0^{-1}(\pi_1 - \pi_0\nu_1 - \pi_0\pi_1^{-1}\beta_1\bar{\nu}_1) \tag{63d}$$

$$b_1 = \alpha_0^{-1}\beta_1(\pi_1 - \pi_0\pi_1^{-1}\beta_1\bar{\nu}_1). \tag{63e}$$

Similarly from eq 53a we obt

$$T_1 = -\alpha_0(a_0 + a_1\delta) \tag{63f}$$

$$a_0 = -\alpha_0^{-1}\pi_0\pi_1^{-1}\bar{\nu}_1 \tag{63g}$$

$$a_1 = \alpha_0^{-1}(\pi_1 - \pi_0\pi_1^{-1}\bar{\nu}_1\beta_1) \tag{63h}$$

where $\bar{\nu}_1 = \bar{\nu}(T_1)$.

Using eq 63b, we will reduce eq 63a to

$$f_{10} = -K_{11}P_1'(n_1^+) + bK_{21}\alpha_0. \tag{64a}$$

Neglecting terms representing sensitive heat, we will reduce eq 49 to

$$k_0\alpha_0 + Lf_{10} = k_{21}\alpha_1 - \rho_{30}V_0L(w_0 - \nu_1). \tag{64b}$$

Now we will recall a special case studied (Nakano 1990) where V_0 vanishes and the steady growth of an ice layer occurs. In such a case, π_0 vanishes and eq 63c is reduced to

$$b = (1+\eta\delta)^{-1}(1+\beta_1\delta) \tag{65a}$$

$$\beta_1 = -k_0^{-1}c_1 f_{10}. \tag{65b}$$

Also eq 64b is reduced to

$$k_0\alpha_0 + Lf_{10} = k_{21}\alpha_1. \tag{65c}$$

It was found (Nakano 1990) that the steady growth of an ice layer occurs under the conditions given as

$$(k_{21}/k_0)\alpha_1 > \alpha_0 \tag{66a}$$

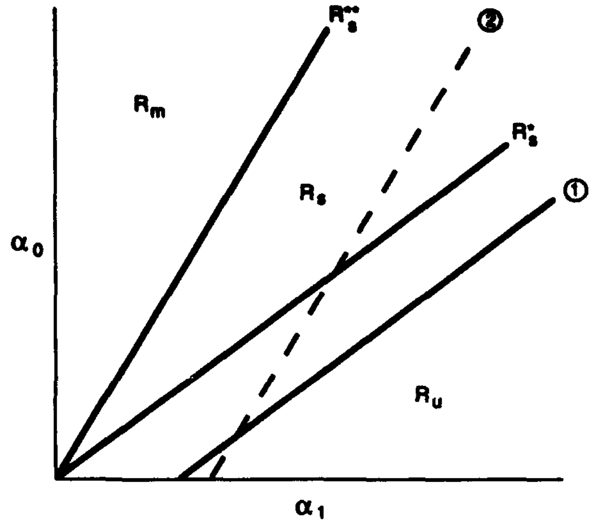

Figure 3. Temperature gradients α_1 and α_0.

$$P_1'(n_1^+) > 0. \tag{66b}$$

Using eq 64a and 65c and combining eq 66a and 66b, we obtain

$$(k_{21}/k_0)\alpha_1 > \alpha_0 > k_{21}(k_0 + LbK_{21})^{-1}\alpha_1 . \tag{66c}$$

The region R_s in Figure 3 satisfies eq 66c and the steady growth of an ice layer occurs in R_s. Line R_s^{**} in Figure 3 is given as

$$\alpha_0 = (k_{21}/k_0)\alpha_1. \tag{67a}$$

Line R_s^{**} is the boundary between R_s and R_m where an ice layer melts. The boundary R_s^* in Figure 3 is given as

$$\alpha_0 = k_{21}(k_0 + Lb^*K_{21}^*)^{-1}\alpha_1 \tag{67b}$$

where superscripts * are used to indicate the value of any variable when a point (α_1, α_0) belongs to R_s^*. Since b^* and K_{21}^* generally depend on α_0 and α_1, the boundary R_s^* between R_s and R_u, where the steady growth of an ice layer does not occur, is a curve stemming from the origin. From eq 66b we obtain

$$P_1'(n_1^+) > 0 \qquad \text{in } R_s \tag{67c}$$

$$P_1'(n_1^+) = 0 \qquad \text{on } R_s^*. \tag{67d}$$

Now we will examine the case in which V_0 is positive and the steady growth of ice-rich frozen soil occurs. First we will show that the necessary condition for the steady growth is given as

$$P_1'(n_1^+) \geq 0. \tag{68}$$

Suppose that eq 68 does not hold true. Since $P_1'(\xi) < 0$ in a neighborhood of n_1 in R_1, there exists a point ξ_1 in this neighborhood such that $P_1(\xi_1) > P_{21}$. Also $P_{21} > P_{10}$ because $P_{21} \geq 0$ and $f_{10} > 0$. Since $P_1(\xi)$ is continuous in R_1, there exists at least one point ξ_2 with $\xi_1 > \xi_2 > n_0$ such that $P_1(\xi_2) = P_{21}$. This implies that another ice-rich frozen soil layer can exist in R_1. This is obviously contradictory to our assumption that there is no ice-rich frozen soil layer in R_1.

When eq 68 holds true, from eq 64a and 64b we obtain

$$\alpha_0 \geq k_{21}b_2^{-1}\alpha_1 - b_3 \tag{69a}$$

where b_2 and b_3 are defined as

$$b_2 = k_0 + LbK_{21} \tag{69b}$$

$$b_3 = b_2^{-1}\rho_{30}V_0L(w_0 - v_1). \tag{69c}$$

In Figure 3 we will draw curve 1 given as

$$\alpha_0 = k_{21}b_2^{-1}\alpha_1 - b_3 . \tag{69d}$$

Since $V_0 > 0$, curve 1 must be in R_u and converges to R_s^* when V_0 approaches zero. When V_0 vanishes and the steady growth of an ice layer occurs, a line of constant f_{10} is parallel to R_s^{**}, such as broken line 2 in Figure 3. It follows from eq 64b that line 2 is still the line of constant f_{10} except in R_u where line 2 is the line of constant $f_{10} + \rho_{30}V_0(w_0 - v_1)$.

It follows from eq 69d that the distance between curve 1 and R_s^* increases with increasing V_0. From eq 34g we find that the ice content in R_2 decreases with increasing V_0. The condition eq 69a implies that the steady growth of frozen soil occurs in the region bounded by curve 1 and R_s^*. Since V_0 is an arbitrary positive number, eq 69a also implies that the steady growth of frozen soil occurs everywhere in R_u. However, the steady growth of ice-rich frozen soil is anticipated to occur in the part of R_u not far from the boundary R_s^*.

Suppose that a point (α_1, α_0) in R_u is given; then we can find V_0 that satisfies eq 69d. At the end of the preceding section we had three independent variables, α_0, α_1 and V_0. Since these three variables are related by eq 69d, we now have only two independent variables, α_0 and α_1. In other words, we have found that the condition of the steady growth of ice-rich frozen soil is uniquely determined by two independent variables, α_0 and α_1,

under a given hydraulic condition and overburden pressure. We have shown that there exists a traveling wave solution containing two independent parameters, α_0 and α_1, to the problem of steadily growing ice-rich frozen soil.

DISCUSSION

One of the outstanding questions among researchers of frost heave has been the relationship between the rate of frost heave r and the speed of a frost front V_0. A significant amount of effort has been made to determine empirically this relationship under the hypothesis that r is uniquely determined by V_0. The empirical relationships between r and V_0 reported in the literature sometime disagree with (or even contradict) each other (Takashi et al. 1978), and there appears to be no consensus among researchers. This situation casts a serious doubt upon the validity of the underlying hypothesis that r is uniquely determined by V_0.

We will show that r is not uniquely determined by V_0 if M_1 and the assumptions used in our analysis are valid. Using eq 64b, we will reduce eq 34f to

$$d_2 r = L^{-1}(k_{21}\alpha_1 - k_0\alpha_0) - \rho_{30} V_0\left(d_1^{-1} d_2 w_0 - v_1\right). \tag{70a}$$

Since α_1 and α_0 are related by eq 69d, we can express r as a function of V_0 and either α_0 or α_1. We will write eq 70a in two ways as

$$d_2 r = S_1\alpha_0 + S_2 \tag{70b}$$

$$= S_3\alpha_1 + S_4 \tag{70c}$$

where S_i is given as

$$S_1 = bK_{21} \tag{70d}$$

$$S_2 = \left(1 - d_1^{-1} d_2\right)\rho_{30} V_0 w_0 \tag{70e}$$

$$S_3 = bK_{21} k_{21} b_2^{-1} - \left(1 - d_1^{-1} d_2\right) v_1\big]\rho_{30} V_0 \tag{70f}$$

$$S_4 = -\left[\left(d_1^{-1} d_2 - k_0 b_2^{-1}\right)(w_0 - v_1)\right. \tag{70g}$$

We will examine the value of S_i for a special case in which σ vanishes and V_0 is much less than f_{10}. The limiting value of S_i as V_0 approaches zero is given as

$$S_1 = b^* K_{21}^* \tag{71a}$$

$$S_2 = 0 \tag{71b}$$

$$S_3 = b^* K_{21}^* S \tag{71c}$$

$$S_4 = 0 \tag{71d}$$

$$S = k_{21}\left(k_0 + L b^* K_{21}^*\right)^{-1}. \tag{71e}$$

It follows from eq 71a, 71b, 71c and 71d that the last term on the right-hand side of eq 70b (or 70c) is negligible in comparison with the first term in the right-hand side of eq 70b (or 70c) when V_0 is much less than f_{10} and $\alpha_i \geq 1.0$ °C cm^{-1} for $i = 1,2$. It is easy to find that S_1, S_2 and S_3 are positive, but the sign of S_4 is not certain.

According to the results of our analysis, we have found that the rate of frost heave r is not uniquely determined by the speed of a frost front V_0 alone and that the relationship between r and V_0 strongly depends on α_i ($i = 0,1$). To examine the validity of eq 70b we will use reported experimental data. Takashi et al. (1978) conducted a series of frost heave tests in which the temperature of the unfrozen part R_0 was kept constant at 0.2–0.3°C higher than the freezing point of a sample so that the speed V_0 of a frost front n_0 was kept nearly constant. Dividing eq 70b by $d_2 V_0$, we will reduce eq 70b to

$$\hat{\xi} = rV_0^{-1} = d_2^{-1} bK_{21}\alpha_0 V_0^{-1} + \left(d_2^{-1} - d_1^{-1}\right)\rho_{30} w_0. \tag{72}$$

Takashi et al. (1978) called $\hat{\xi}$ the "frost heave ratio."

Analyzing their data, Takashi et al. (1978) found empirically that $\hat{\xi}$ is uniquely determined by V_0 for a given applied pressure σ. A typical behavior of $\hat{\xi}$ vs. V_0 obtained by them is reproduced in Figure 4 where a curve is drawn that approximately represents their data points taken with their sample 2 under the applied pressure σ = 304 kPa. In their tests the temperature profile in the sample was not measured and it is difficult to assess the variability of α_0. However, if α_0 is kept nearly constant and K_{21} mainly depends on $\alpha_0 V_0^{-1}$, then their empirical relationships between $\hat{\xi}$ and V_0 are consistent with eq 72 as we will show below.

Since the second term on the right side of eq 72 is a given constant, $\hat{\xi}$ approaches asymptotically this constant as V_0 becomes infinite. The value of $\hat{\xi}$ increases with the decreasing V_0 until $\hat{\xi}$ becomes infinite when V_0 vanishes and an ice layer grows.

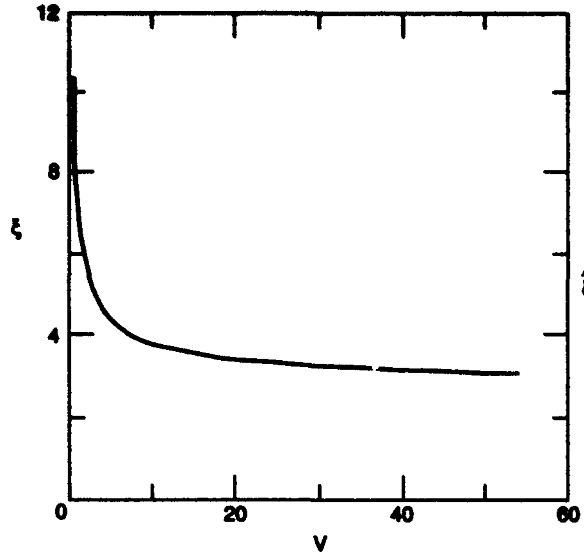

Figure 4. Frost heave ratio $\hat{\xi}$ (%) vs. V_0 (cm d^{-1}) obtained empirically by Takashi et al. (1978).

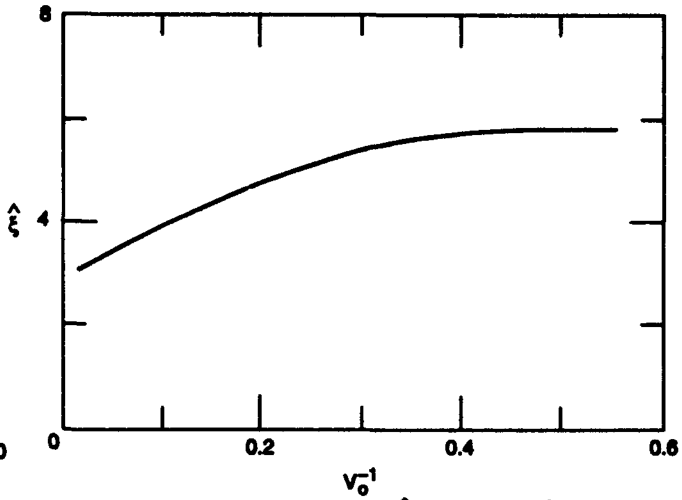

Figure 5. Frost heave ratio $\hat{\xi}$ (%) vs. V_0^{-1} (cm^{-1} d) obtained empirically by Takashi et al. (1978).

The curve $\hat{\xi}$ vs. V_0 in Figure 4 is converted into the curve $\hat{\xi}$ vs. V_0^{-1} in Figure 5. It is clear from eq 72 that the gradient of the curve in Figure 5 is proportional to K_{21} if eq 72 holds true. From Figure 5 we find that the gradient of the curve tends to decrease with the increasing V_0^{-1}; namely, K_{21} is a decreasing function of V_0^{-1}.

We have derived eq 72 under the assumption that the speed of a frost front V_0 is constant. Therefore, eq 72 is not anticipated to hold true for the transient freezing in which V_0 varies with time. However, eq 72 may approximately hold true for the transient case in which the change of V_0 with time is small. Analyzing the data on transient freezing tests obtained by Akagawa (1990), Miyata and Akagawa (1991) empirically found that $\hat{\xi}$ may be uniquely determined by $\alpha_o V_0^{-1}$, though data are limited. Their data $\hat{\xi}$ of two tests (test A with $\sigma = 60$ kPa and test B with $\sigma = 110$ kPa) vs. $\alpha_o V_0^{-1}$ are presented in Figure 6 where a curve is drawn to show the trend of the data points.

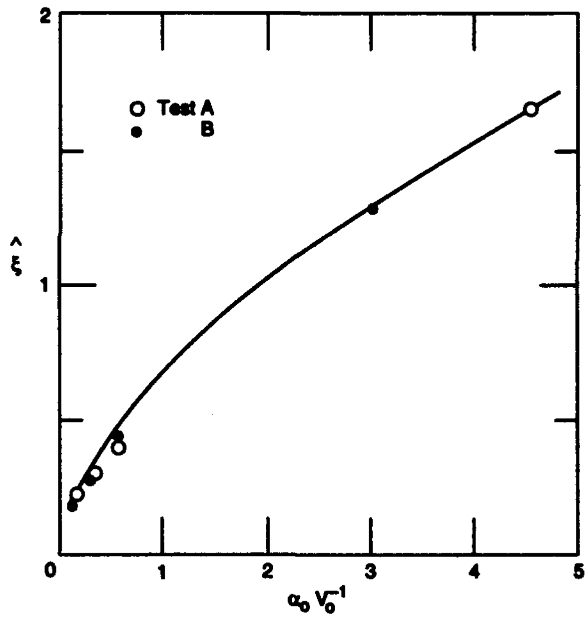

Figure 6. Frost heave ratio $\hat{\xi}$ (%) vs. $\alpha_0 V_0^{-1}$ (°C cm^{-2} d) obtained empirically by Miyata and Akagawa (1991).

A soil specimen of 8.5-cm length was frozen from the bottom up with constant boundary temperatures (Akagawa 1990). The data points ($\hat{\xi}$, $\alpha_o V_0^{-1}$) were taken during the time period from 4 to 45 hours after the start of the test. The speed V_0 decreased and α_0 increased monotonically with time; hence, the value $\alpha_o V_0^{-1}$ increased monotonically with time. In test A, for instance, V_0 changed from 9.12 cm d^{-1} at 4 hours to nearly zero at 45 hours while α_0 changed from about 0.6°C cm^{-1} at 4 hours to about 1.5°C cm^{-1} at 23 hours. It is quite interesting that eq 72 may hold true despite such rates of change in V_0 and α_0 as described above. From Figure 6 we find a trend similar to that of Figure 5: the gradient of the curve $\hat{\xi}$ vs. $\alpha_o V_0^{-1}$ tends to decrease with the increasing $\alpha_o V_0^{-1}$, namely, K_{21} is a decreasing function of $\alpha_o V_0^{-1}$.

We have studied the steady growth of ice-rich frozen soil by using M_1. We have shown that there exists a traveling wave solution to the problem of steadily growing ice-rich frozen soil and that this solution is reduced to the solution to the problem

of a steadily growing ice layer when the velocity $\dot{n}_0$ vanishes. We have also shown that the steady growth condition of ice-rich frozen soil under given hydraulic conditions and applied pressures is uniquely determined by a set of two physical variables, α_0 and α_1. We will present the results of our experimental study in another report.

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APPENDIX A: EXACT SOLUTION OF EQUATION 45

When $k_1(\xi)$ is given by eq 40e, we will introduce a new independent variable X defined for $n_1 \geq \xi \geq n_0$ as

$$X = 1 + \eta(\xi - n_0), \qquad 1 + \eta\delta \geq X \geq 1. \qquad \text{(A1)}$$

Using X, we will reduce eq 45 to

$$\frac{dT}{dX} = (\eta X)^{-1}\left(\beta_1 T - \alpha_0 - k_0^{-1}\Lambda L\right). \qquad \text{(A2)}$$

Multiplying eq A2 by $X^{-(\beta_1/\eta)}$, we will write eq A2 as

$$\frac{d}{dX} T X^{-(\beta_1/\eta)} = -(\eta X)^{-1} X^{-(\beta_1/\eta)}\left(\alpha_0 + k_0^{-1}\Lambda L\right). \qquad \text{(A3)}$$

Suppose that a solution $T(X)$ of eq A2 exists. Since the right side of eq A3 is negative, the function $TX^{-(\beta_1/\eta)}$ is a decreasing function of X. Therefore, $T(X)$ must be negative for $X > 1$ because $T = 0$ at $X = 1$. We have found that a solution $T(X)$ is negative for $X > 1$ if it exists.

Integrating eq A2, we obtain

$$\ln X = \int_0^T \eta\left(\beta_1 T - \alpha_0 - k_0^{-1}\Lambda L\right)^{-1} dT. \qquad \text{(A4)}$$

Since the integrand of eq A4 is continuous, a solution $T(X)$ exists. It is easy to see that the right side of eq A2 satisfies a Lipschitz condition with respect to T because the function $\Lambda(T)$ possesses a continuous first derivative as assumed. Based on an elementary theorem of ordinary differential equations (Sansone and Conti 1964), we may conclude that eq A2 has a unique negative solution $T(X)$ for $X > 1$.

It follows from eq A2 that the unique solution of eq A2 is decreasing with x (or ξ) because the right side of eq A2 is negative. We have shown that eq 45 has a unique and decreasing solution for $n_1 \geq \xi \geq n_0$.

APPENDIX B: APPROXIMATE SOLUTION OF EQUATION 45

As we have shown above (App. A), eq 45 has a unique and decreasing solution. This implies that X (or ξ) and T are one-to-one. Treating Λ as a function of X, we will write eq A2 as

$$T(X) = \pi_1 \beta_1^{-1}\left(1 - X^{\beta_1\eta^{-1}}\right) + \pi_0 \eta^{-1} Y X^{\beta_1\eta^{-1}} \quad (B1)$$

where π_0, π_1, and Y are defined as

$$\pi_0 = k_0^{-1} L \rho_{30} V_0 \quad (B2)$$

$$\pi_1 = \alpha_0 + \pi_0 w_0 \quad (B3)$$

$$Y(X) = \int_1^X \nu\, X^{-\left(1+\beta_1\eta^{-1}\right)} dX. \quad (B4)$$

We will rewrite eq A2 as

$$\frac{dT}{dX} = -(\eta X)^{-1} \pi_1 Y_1 \quad (B5)$$

where Y_1 is defined as

$$Y_1 = 1 - \pi_1^{-1}(\beta_1 T + \pi_0 \nu). \quad (B6)$$

Using eq B5, we will reduce eq B4 to

$$Y = \eta \pi_1^{-1} \bar{\nu} \quad (B7)$$

$$\bar{\nu} = -\int_0^T \nu\, X^{-\beta_1\eta^{-1}} Y_1^{-1} dT. \quad (B8)$$

Using eq B7, we will reduce eq B1 to:

$$T(\xi) = \pi_1 \beta_1^{-1}\left(1 - X^{\beta_1\eta^{-1}}\right) + \pi_0 \pi_1^{-1} \bar{\nu} X^{\beta_1\eta^{-1}}. \quad (B9)$$

Using eq B5 and B9, we obtain:

$$T'(\xi)X = -X^{\beta_1\eta^{-1}}\left(\pi_1 - \pi_0\pi_1^{-1}\beta_1\bar{\nu}\right) + \pi_0 \nu. \quad (B10)$$

We will seek approximate equations for eq B9 and B10 when the following condition holds true:

$$\eta\delta < 1 \quad \text{and} \quad \beta_1\delta < 1. \quad (B11)$$

We found that the condition, eq B11, holds true when the steady growth of an ice layer occurs (Takeda and Nakano 1990). When eq B11 holds true, we may use an approximation (Nakano 1990) given as:

$$X^{\beta_1\eta^{-1}} = 1 + \beta_1(\xi - n_0). \quad (B12)$$

Using eq B12, we will reduce eq B9 and B10 to

$$T(\xi) = \pi_0\pi_1^{-1}\bar{\nu} - \left(\pi_1 - \pi_0\pi_1^{-1}\bar{\nu}\,\beta_1\right)(\xi - n_0) \quad (B13)$$

$$T'(\xi)X = -\pi_1 + \pi_0\left(\nu + \pi_1^{-1}\beta_1\bar{\nu}\right) - \beta_1\left(\pi_1 - \pi_0\pi_1^{-1}\beta_1\bar{\nu}\right)(\xi - n_0). \quad (B14)$$

For the calculation of $\bar{\nu}$, $X^{-\beta_1\eta^{-1}}$ is approximated by using eq B12 as

$$X^{-\beta_1\eta^{-1}} = [1 + \beta_1(\xi - n_0)]^{-1}. \quad (B15)$$

When V_0 is very small, from eq B13 $T(\xi)$ is given as

$$T(\xi) = -\pi_1(\xi - n_0). \quad (B16)$$

Substituting eq B16 into eq B15 we obtain

$$X^{-\beta_1\eta^{-1}} = \left(1 - \beta_1\pi_1^{-1}T\right)^{-1}. \quad (B17)$$

Now $\bar{\nu}$ is approximated as

$$\bar{\nu} = -\int_0^T \nu\left[Y_1\left(1 - \beta_1\pi_1^{-1}T\right)\right]^{-1} dT. \quad (B18)$$

APPENDIX C: COMPUTATION OF I_1

Using eq B5, we will reduce eq 56b to

$$-\pi_1 K_{10} I_1 = \int_0^T X(K_{10}/K_1)Y_1^{-1} dT. \qquad \text{(C1)}$$

The term $\beta_1 T$ in eq B6 describes the effect of sensitive heat that is much less than π_1. Also the term $\pi_1^{-1}\pi_0 \nu$ in eq B6 is less than one. Therefore, the term $\pi_1^{-1}(\beta_1 T + \pi_0 \nu)$ is generally less than one and we may approximate Y_1^{-1} as

$$Y_1^{-1} = 1 + \pi_1^{-1}(\beta_1 T + \pi_0 \nu). \qquad \text{(C2)}$$

Using eq 55b, 55c and C2, we will reduce eq C1 to:

$$-\pi_1 K_{10} I_1 - w_o \pi_0 \pi_0^{-1} T_1 \phi_{21}$$

$$= \int_0^T \dot{\phi}_1 T_1 (1 + \pi_1^{-1} \beta_1 T) dT$$

$$= \phi_{11} T_1 (1 - \varepsilon) \qquad \text{(C3)}$$

where

$$\dot{\phi}_1 = d\phi_1 / dT$$

$$\varepsilon = \beta_1 \pi_1^{-1} \left(- T_1 + \phi_{11}^{-1} \int_0^{T_1} \phi_1 dT \right). \qquad \text{(C4)}$$

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13. ABSTRACT (*Maximum 200 words*)

The results of mathematical and experimental studies presented in preceding reports clearly show that the model M_1 accurately describes the properties of a frozen fringe when the steady growth of an ice layer occurs. In this work the steady growth of ice-rich frozen soil is studied by using M_1. Deriving a traveling wave solution to the problem, we have found that the condition of steady growth of ice-rich frozen soil is uniquely determined by a set of two physical variables, such as α_0 and α_1 used earlier, under given hydraulic conditions and overburden pressures and that the traveling wave solution converges to the solution to the problem of a steadily growing ice layer when the velocity of the 0°C isotherm relative to the unfrozen part of the soil vanishes.

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